We will calculate how the output power of a Michelson interferometer using DC readout responds to both length changes (eg from a gravitational wave) and to fluctuations in the laser frequency (frequency noise). Consider a Michelson Interferometer with arms of length L_1 and L_2, and an input laser power of P_{in}. We define the differential and common arm lengths as: $L_d = L_1 - L_2$ and $L_c = L_1 + L_2$, equivalent to CARM and DARM in LIGO. The original arm lengths are then:

$$L_1 = \frac{L_c + L_d}{2} \qquad \text{and} \qquad \mathrm{L_2} = \frac{\mathrm{L_c - L_d}}{2} \tag{1}$$

When calculating the response of an interferometer, experimentalists often use some shorthand notation where the electric field amplitude is integrated over space and is normalised such that $P = E^*E$. The units are now $\sqrt{W}$, and the phase can be related to the phase of a co-propagating plane wave. Ignoring the time dependence of the electric field (all time terms are equal in this case), we can write our input electric field from the laser as: $E_{\mathrm{in}}e^{ikL}$, where $E_{in} = \sqrt{P_{in}}$, the wavenumber k is related to the laser wavelength and frequency by $k = 2\pi/\lambda = 2\pi f/c$, and L is the optical path length starting from the beamsplitter. Using the convention that each transmission through a mirror gives a phase shift of i, and assuming perfectly reflective end mirrors, we find that the electric field at the interferometer output is given by:

$$E_{out} = irtE_{in}e^{i2kL_1} + irtE_{in}e^{i2kL_2} \tag{2}$$

where $r = \sqrt{R}$ and $t = \sqrt{T}$ are the (amplitude) reflectivity and transmissivity of the beamsplitter. Separating this into common and differential lengths, we find:

$$E_{out} = irtE_{in}e^{ikL_c}(e^{ikL_d} + e^{-ikL_d}) \tag{3}$$

The output power of the our interferometer is found by taking the modulus squared of the electric field (with our shorthand notation):

$$\begin{aligned} P_{out} = E^*_{out}E_{out} &= r^2t^2P_{in}(2 + e^{i2kL_d} + e^{-i2kL_d}) && (4)\\ &= \frac{P_{in}}{2}(1 + \cos(2kL_d)) && (5)\end{aligned}$$

where we've assumed $r^2 = t^2 = 0.5$. We can see here that there is *no dependence on the common arm length.*

We can safely assume that fluctuations in laser frequency, δf, are much (much) smaller than the laser frequency, f_0. We can also assume that microscopic changes in arm length, δL_d, for example from 'detuning' or a gravitational wave, are very small compared with the static arm length difference L_{d0}. In LIGO the arm-length difference is set to about 20 cm for technical reasons. Combining these:

$$\begin{aligned} kL_d &= (k_0 + \delta k)(L_{d0} + \delta L_d) && (6)\\ &\approxeq k_0L_{d0} + k_0\delta L_d + \delta kL_{d0} && (7)\end{aligned}$$

The leading term, k_0L_{d0}, defines both the macroscopic length and operating condition of our interferometer. The second term is the 'signal', a phase change caused by mirror displacement, and the third is frequency noise. For operation on a perfect 'dark port', this leading term is set to $(m + 1/2)\pi$. This is a problem since the output will be quadratic with length. Instead, the interferometer is offset by a small 'detuning', β, where $(k_0\delta L_d + \delta kL_{d0}) << \beta << \pi$ and $k_0L_{d0} = (m + 1/2)\pi + \beta$, giving:

$$\begin{aligned} P_{out} &= \frac{P_{in}}{2}[1 - \cos(2(\beta + k_0\delta L_d + \delta kL_{d0}))] && (8)\\ &\approxeq P_{in}(\beta^2 + 2\beta(k_0\delta L_d + \delta kL_{d0})) && (9)\\ &\approxeq P_{in}(\beta^2 + 2\beta k_0\delta L_d + \frac{4\beta\pi\delta fL_{d0}}{c}). && (10)\end{aligned}$$

We can clearly see that frequency fluctuations couple directly into the output power (our DARM channel), and that they increase relative to the signal term proportional to the static arm length difference.