Pole and zeroes are used to define transfer functions. They are either purely real or a complex-conjugate pair, and they are defined in terms of the damping factor and resonance frequency. The only difference between a pole and a zero is whether they appear in the numerator (zeroes) or denominator (poles).

Defining poles and zeroes

A pole can be real, or it can be complex. The magnitude of a real pole is just the corner (angular) frequency of, eg, a low-pass filter, and it is only stable for negative values. A complex pole is defined as

$$p = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \tag{1}$$

where ω_0 is the (angular) frequency of the pole, and the damping factor α is related to the quality factor by

$$Q = \frac{\omega_0}{2\alpha} \tag{2}$$

$$\alpha = \frac{\omega_0}{2Q}. \tag{3}$$

Note that for $\omega_0 > \alpha$ the pole is $under - damped$ and forms a conjugate pair. For $\omega_0 < \alpha$ the pole is $over - damped$ and there are two (different) real-valued poles.

To extract the frequency and Q from a pole

$$\omega_0 = \sqrt{\mathrm{Re}(p)^2 + \mathrm{Im}(p)^2} = |p| \tag{4}$$

$$Q = \frac{\omega_0}{-2\,\mathrm{Re}(p)}. \tag{5}$$

Finally, to define a pole (pair) directly from frequency and Q

$$p = -\frac{\omega_0}{2Q} \pm \sqrt{(\frac{\omega_0}{2Q})^2 - \omega_0^2}. \tag{6}$$

Calculating the transfer function

With poles (and zeroes) defined, the total transfer function is calculated using

$$H(s) = K\frac{(s - z_1)(s - z_2)...(s - z_n)}{(s - p_1)(s - p_2)...(s - p_n)} \tag{7}$$

where z_n are the zeroes, p_n are the poles, K is the proportional gain, and s is the *complex number frequency* defined by $s = \sigma + i\omega$. To calculate the steady-state transfer function, simply let $\sigma = 0$, or explicitly

$$H(\omega) = K\frac{(i\omega - z_1)(i\omega - z_2)...(i\omega - z_n)}{(i\omega - p_1)(i\omega - p_2)...(i\omega - p_n)}. \tag{8}$$

Note that poles are not normalised. To get unit response at zero frequency from a single pole (ie $H(0) = 1$), the proportional gain needs to be set to the $K = |p|$. For a fully 'normalised' transfer function

$$K = \frac{|p_1|\,|p_2|...|p_n|}{|z_1|\,|z_2|...|z_n|}. \tag{9}$$

In the time domain

This section is loose thoughts. Do not use as reference.

Poles have a time domain response something like

$$Ae^{i\phi}e^{pt}.$$

For a conjugate pair, this looks something like

$$Ae^{\mathrm{Re}(p)t}\sin(\mathrm{Im}(p)t+\phi)$$

where A and ϕ are determined by initial conditions. We can see that any pole with a positive real component will make the whole system unstable, and that for negative real components, there will be an exponentially decaying term. For the conjugate pair, this is a decaying sinusoid sinusoid at (angular) frequency $\mathrm{Im}(p)$. Note that the sinusoid is not at frequency ω_0, the damping 'pulls' the resonant frequency away from the perfect, undamped resonance.