

Guided Waves

To better understand guided waves, we will look at the wave solutions to Maxwell's equations. In a (charge-free, current-free) vacuum:

$$\textcircled{1} \quad \vec{\nabla} \cdot \vec{E} = 0 \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \textcircled{2}$$

$$\textcircled{3} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \textcircled{4}$$

taking the curl of $\textcircled{2}$,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\text{but } \vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} \quad \textcircled{1} \quad = -\nabla^2 \vec{E}$$

$$\therefore \boxed{\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}} \quad \left(= \frac{1}{c_0^2} \frac{\partial^2 \vec{E}}{\partial t^2} \right)$$

$$\text{Similarly, } \boxed{\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}} \Rightarrow \text{electromagnetic wave equation.}$$

Side note:

The general solution is a superposition of waves of the form:

$$\vec{E}(\vec{r}, t) = g(wt - \vec{k} \cdot \vec{r})$$

$$\vec{B}(\vec{r}, t) = g(wt - \vec{k} \cdot \vec{r})$$

Due to linearity, any wave can be decomposed into (Fourier) sinusoids,

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \cos(wt - \vec{k} \cdot \vec{r} + \phi_0) \quad (\text{similar for } \vec{B})$$

and our special case, linear-polarised, $+z$ propagating plane wave is:

$$\vec{E} = E_0 \cos(wt - kz) \hat{x} \quad (\text{or } g)$$

(22a)

E-M plane-waves

As revision, we know that there is a wave-equation that describes electromagnetic radiation that results directly from Maxwell's equations (see, e.g., wikipedia 'Electromagnetic wave equation'). One useful and relevant solution is a plane-wave:

$$\vec{E}(x, y, z, t) = E_0 \hat{e}_p \cos(\omega t - \vec{k} \cdot \vec{r} + \phi)$$

amplitude (V/m) propagation wave-vector, $k = \frac{2\pi}{\lambda}$
 ↓ ↓
 polarisation frequency = $2\pi\nu$ (angular)
 ↑ ↑
 phase

For now, we can ignore the polarisation and the absolute phase (which only matters when comparing light fields). If we also, for now, only consider light propagating along the $+z$ direction, we get:

$$E(z, t) = E_0 \cos(\omega t - kz)$$

$$= \frac{E_0 e^{i(\omega t - kz)}}{2} + C.C.$$

or $E = \text{Re}(E')$, where $E' = E_0 e^{i(\omega t - kz)}$.

In interferometer calculations, we often want to know the field at one point in time, usually in a steady-state, allowing us to neglect the time term, and we're left with:

$$E = E_0 e^{-ikz}$$

Note that this is not a physical quantity!

Detecting light

Light is too fast, and we cannot easily see the electric field, but we can definitely feel the flow of energy, or the Poynting vector:

$$\vec{S} = \vec{E} \times \vec{H} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \quad (\text{in vacuum})$$

For our plane-wave, this simplifies to:

$$|\vec{S}| = \frac{1}{\mu_0 c} E^2 = C \epsilon_0 E_0^2 \cos^2(\omega t) = \frac{C \epsilon_0 E_0^2}{2} (1 + \cos(2\omega t))$$

↑
observing from a fixed point in space.

The instantaneous power fluctuates at 2ω - much too fast (that's more than 10^{14} cycles/second!), so we see the intensity, or the average Poynting vector:

$$\langle |\vec{S}| \rangle = \frac{C \epsilon_0 E_0^2}{2} = \frac{C \epsilon_0 E \cdot E^*}{2}$$

(denotes time-average)

For a laser beam, where we can measure 100% of the power, we use a simpler, more intuitive definition of the amplitude that incorporates the constants, and integrates over the transverse profile, resulting in:

$$P = E E^*$$

Where this new E has units of \sqrt{W} , and it is very far from the original electric field definition!

See pages 8, 9, and 32 in "interferometer techniques for gravitational wave detection".

Intensity (continued)

using this short-hand, let's take a laser beam with power P_0 . Its amplitude is:

$$E = \sqrt{P_0} e^{-ikz}$$

Measuring at $z = 0$,

$$P = E \cdot E^* = \sqrt{P_0} \sqrt{P_0} = P_0.$$

Measuring at $z \Rightarrow kz = \phi$,

$$P = \sqrt{P_0} e^{-i\phi} \sqrt{P_0} e^{i\phi} = P_0,$$

and as expected the power doesn't depend on phase.

Spatial dependence

In this course, we are often concerned with spatial distributions of intensity. Carrying all the constants through is traumatic and we run into problems with infinities, both in distance and transverse extent. For this reason, we invoke the magical 'proportional to' sign, and consider the relative distribution of intensity.

Note that intensity is distributed [W/m^2] and power is integrated [W].

For the spatial dependence calculations, we start with a z-propagating beam and determine the small-angle off-axis behaviour.